SOME REMARKS ON THE ENERGY NORM AND Z–Z ERROR ESTIMATOR

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In the present paper some details of the problem of *a posteriori* error estimates are reconsidered. It has been shown that for smoothed solutions, at variance with raw finite element solution, the error in energy is *not* equal to the energy in error. As a consequence, it is dubious to speak about 'superconvergence' of recovered derivatives. Nevertheless, it has been found that so–called Z–Z error estimator (error in energy of smoothed with respect to raw finite element solution) is a useful procedure for estimation of the error of the raw finite element solution, which also can be further improved.

1. INTRODUCTION

There is a numerical evidence that, at least for four noded isoparametric elements, any stress recovery procedure is less accurate in *strain energy* than direct FEA (Finite Element Analysis). There are two general classes of the stress smoothing procedures [1,2]. If carried out over a whole finite element mesh, the procedure is known as a global smoothing. Local smoothing is performed at each node or small group of nodes, per instance by averaging of the stresses from neighbouring elements at a particular common node.

The main disadvantage of the FE displacement approach (based on the theorem of minimum potential energy) is that calculated stresses are generally discontinuous at the element interfaces. It simply means that, instead of the unique value of the stress at the global node, we have as many different stress values as there are elements connected. Furthermore, the use of low order elements results in a low order discontinuous approximation. As a consequence, the stress accuracy changes from point to point within an element. As an attempt to overcome the problem of interpretation of the results of numerically discontinuous model of a physically continuous system and improve the overall stress results, the number of different techniques have been proposed. One of the earliest

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attempts to obtain the smoothed stress picture of the model has been the averaging of the nodal stresses of all elements connected at a common node. This is a simple and fast procedure, but numerous examples were reported where cannot be recommended [1]. Nevertheless, from the contemporary point of view it is a classical technique, widely adopted as a reference procedure in numerical examinations. Next, the procedure called "consistent conjugate stress calculation" [2] was introduced in 1971. This method "*...is based on the idea of consistent stress approximation and it approximates such stresses using the notion of a domain influence of the stress intensity at a nodal point*". This is a global stress smoothing method resulting in a set of linear simultaneous equations having well conditioned and positive definite matrix.

Some times later, in 1987, Zienkiewicz and Zhu have been shown that 'smoothing' procedures and *a posteriori* error estimation are closely related, and proposed so–called Z–Z error estimator, based on smoothing procedures.

In the present paper, after a wider analysis of a class of smoothing procedures, we will reconsider two of classical and typical approaches, simple averaging and L_2 projection, and discuss their merits in error estimation and postprocessing fields.

2. PRELIMINARIES

We let $\Omega = \mathrm{R}^n$, where $n = 2$ or 3 denote an open bounded Lipschitzian domain with piecewise smooth boundary $\partial\Omega$. In the problems considered here, working in R^2 rather than in R^3 is not really restrictive and extensions are generally straightforward. Hence we can present our examples in a two–dimensional setting for the sake of simplicity.

The classical equations governing equilibrium of a material body occupying a region Ω are,

$$\begin{aligned} &\mathbf{div}\ \boldsymbol{T} + \boldsymbol{f} = 0 && \text{in } \Omega, \\ &\boldsymbol{u} = 0 && \text{on } \Gamma_{\mathrm{D}}, \\ &\boldsymbol{T}(\boldsymbol{u}) \bullet \boldsymbol{n} = \boldsymbol{t} && \text{on } \Gamma_{\mathrm{N}}, \end{aligned} \tag{1}$$

where $\boldsymbol{T}$ is the symmetric stress tensor while $\boldsymbol{f}$ is the vector of body forces, $\boldsymbol{u}$ denotes a displacement vector, $\boldsymbol{n}$ a unit exterior normal to a boundary. Furthermore, Γ_{D} and Γ_{N} are the Dirichlet and Neumann portions of the boundary $\partial\Omega$, and $\boldsymbol{w}$ and $\boldsymbol{t}$ are the displacements and tractions prescribed on these portions respectively.

The strain–displacement relations and the constitutive equations are

$$\begin{aligned} &2\boldsymbol{e}(\boldsymbol{u}) = \nabla \boldsymbol{u} - \nabla \boldsymbol{u}^{\mathrm{T}}, \\ &\boldsymbol{T}(\boldsymbol{u}) = \boldsymbol{C}\,\boldsymbol{e}\ , \end{aligned} \tag{2}$$

where $\boldsymbol{C}$ is the elasticity tensor. We define $V(\Omega) = \{\boldsymbol{v} \in (H^1(\Omega))^n \colon \boldsymbol{v} = 0 \text{ on } \Gamma_{\mathrm{D}}\}$ where H^1 is, as usual, the space of all functions having square integrable gradients. Then, the variational formulation of the boundary value problem (1) is

$$\begin{aligned} &\text{Find } \boldsymbol{u} \in V(\Omega) \text{ such that } \forall\, \boldsymbol{v} \in V(\Omega) \\ &a(\boldsymbol{u},\boldsymbol{v}) = \ell(\boldsymbol{v}), \end{aligned} \tag{3}$$

where the bilinear form $a : V(\Omega) \times V(\Omega) \Rightarrow \mathrm{R}$ and the linear form $\ell : V(\Omega) \to \mathrm{R}$ are given by the following expressions:

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_{\Omega} \mathbf{C}\boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{v})\mathrm{d}\Omega,$$
$$\ell(\boldsymbol{v}) = \int_{\Omega} \boldsymbol{f} \bullet \boldsymbol{v}\, \mathrm{d}\Omega + \int_{\Gamma_{\mathrm{N}}} \boldsymbol{t} \bullet \boldsymbol{v}\, \mathrm{d}\Gamma. \tag{4}$$

Note that $a(\boldsymbol{u},\boldsymbol{v}) = 2U(\boldsymbol{v})$, twice the *strain energy*.

3. FINITE ELEMENT MODEL

We next summarize finite element approximations of (4). The domain Ω is covered by finite subdomains Ω_K over which piecewise continuous polynomial approximations $\boldsymbol{u}_h$ are performed. The suitable finite element space will be defined as

$$V_h = \left\{ \boldsymbol{u_h} \in V(\Omega) \;\middle|\; \boldsymbol{u}_h\big|_K \in \mathsf{P}_{p_K}(\Omega_K),\ \forall \Omega_K \in \Omega \right\}. \tag{5}$$

The restrictions of the finite element approximation $\boldsymbol{u}_h$ to an element Ω_K belong to the space P_{p_K} of polynomials of degree p_K over Ω_K. The finite element approximation of (3) obtained in the space V_h is characterized by the discrete problem

$$\text{Find } \boldsymbol{u}_h \in V_h(\Omega) \text{ such that } \forall \boldsymbol{v}_h \in V_h(\Omega)$$
$$a(\boldsymbol{u}_h,\boldsymbol{v}_h) = \ell(\boldsymbol{v}_h). \tag{6}$$

4. ENERGY OF ERROR AND ERROR OF ENERGY

As it has been noted in [3], p. 40, if (3) holds for all $\boldsymbol{v}$, it holds for every $\boldsymbol{v}_h$ in V_h, and substracting (6) the result is

$$a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h) = 0 \qquad \forall \boldsymbol{v}_h \in V_h(\Omega), \tag{7}$$

i.e. the error $\boldsymbol{u}-\boldsymbol{u}_h$ is orthogonal to V_h. Equivalently, with respect to the energy inner product a, $\boldsymbol{u}_h$ is the projection of $\boldsymbol{u}$ onto V_h. It follows from (7) that $a(\boldsymbol{u},\boldsymbol{u}_h) = a(\boldsymbol{u}_h,\boldsymbol{u}_h)$, and the Pythagorean theorem holds: *The energy of the error equals the error in the energy*,

$$a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}-\boldsymbol{u}_h) = a(\boldsymbol{u},\boldsymbol{u}) - a(\boldsymbol{u}_h,\boldsymbol{u}_h). \tag{8}$$

Since the left side is necessarily positive, the strain energy in $\boldsymbol{u}_h$ always underestimates the strain energy in $\boldsymbol{u}$:

$$a(\boldsymbol{u}_h,\boldsymbol{u}_h) \le a(\boldsymbol{u},\boldsymbol{u}). \tag{9}$$

5. STRESS PROJECTION

Using (2b), a from (4) can be rewritten as

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_{\Omega} \boldsymbol{AT}(\boldsymbol{u}){:}\boldsymbol{T}(\boldsymbol{v})\mathrm{d}\Omega, \tag{10}$$

where $\boldsymbol{A} = \boldsymbol{C}^{-1}$ is the elastic compliance tensor. Because we know discontinuous stresses

$$\boldsymbol{T}(\boldsymbol{u}_h) = \boldsymbol{C}\nabla \boldsymbol{u}_h, \tag{11}$$

it is customary to project these onto some suitably chosen continuous finite element space

$$Q_h = \left\{ \boldsymbol{S}_h \in Q(\Omega) \;\middle|\; \boldsymbol{S}_{h|K} \in \mathsf{P}_{p_K}(\Omega_K),\ \forall \Omega_K \in \Omega \right\} \tag{12}$$

The restrictions of the finite element approximation $\boldsymbol{u}_h$ to an element Ω_K belong to the space P_{p_K} of polynomials of degree q_K over Ω_K The projection of $\boldsymbol{T}_h$ onto Q_h is characterized by the discrete problem

$$\begin{aligned}&\text{Find } \boldsymbol{T}_h^L \in Q_h(\Omega) \text{ such that } \forall\, \boldsymbol{S}_h \in Q_h(\Omega)\\ &a(\boldsymbol{T}_h^L, \boldsymbol{S}_h) = a(\boldsymbol{T}_h, \boldsymbol{S}_h).\end{aligned} \tag{13}$$

From (13) it follows that $a(\boldsymbol{T}_h^L, \boldsymbol{T}_h) = a(\boldsymbol{T}_h, \boldsymbol{T}_h)$. and again, analogously to (8)

$$a(\boldsymbol{T}_h - \boldsymbol{T}_h^L, \boldsymbol{T}_h - \boldsymbol{T}_h^L) = a(\boldsymbol{T}_h, \boldsymbol{T}_h) - a(\boldsymbol{T}_h^L, \boldsymbol{T}_h^L), \tag{14}$$

i.e. the *energy of error of the projected* $\boldsymbol{T}_h^L$ *with respect to the raw finite element solution* $\boldsymbol{T}_h$ equals the error in the energy of these solutions. Furthermore, since the left side (14) is necessarily positive

$$a(\boldsymbol{T}_h^L, \boldsymbol{T}_h^L) \le a(\boldsymbol{T}_h, \boldsymbol{T}_h), \tag{15}$$

or, as it has been said by Strang and Fix [3] p. 51, '. . . *projection cannot increase energy*'. It is also simple to show that in the inner product space Q_h function closest to a given $\boldsymbol{T}_h$ is always the projection of $\boldsymbol{T}_h$ onto Q_h. For any $\boldsymbol{R}_h \in Q_h(\Omega)$

$$a(\boldsymbol{T}_h - \boldsymbol{T}_h^L - \boldsymbol{R}_h, \boldsymbol{T}_h - \boldsymbol{T}_h^L - \boldsymbol{R}_h) = a(\boldsymbol{T}_h - \boldsymbol{T}_h^L, \boldsymbol{T}_h - \boldsymbol{T}_h^L) - 2a(\boldsymbol{T}_h - \boldsymbol{T}_h^L, \boldsymbol{R}_h) + a(\boldsymbol{R}_h, \boldsymbol{R}_h)$$

If (13) holds, then

$$a(\boldsymbol{T}_h - \boldsymbol{T}_h^L, \boldsymbol{T}_h - \boldsymbol{T}_h^L) \le a(\boldsymbol{T}_h - \boldsymbol{T}_h^L - \boldsymbol{R}_h, \boldsymbol{T}_h - \boldsymbol{T}_h^L - \boldsymbol{R}_h),$$

or simply

$$a(\boldsymbol{T}_h - \boldsymbol{T}_h^L, \boldsymbol{T}_h - \boldsymbol{T}_h^L) = \inf_{\boldsymbol{S}_h \in Q_h} a(\boldsymbol{T}_h - \boldsymbol{S}_h, \boldsymbol{T}_h - \boldsymbol{S}_h). \tag{16}$$

6. ENERGY ERROR AND ENERGY NORM

The error in energy e_E can be defined as the difference between the energies of exact $\boldsymbol{T}$ and approximate $\boldsymbol{P}$ stress fields,

$$e_E = a(\boldsymbol{T}, \boldsymbol{T}) - a(\boldsymbol{P}, \boldsymbol{P}). \tag{17}$$

The appropriate *energy* (*error*) *norm* $\|e\|_E$ follows from

$$\|e\|_E^2 = |a(\boldsymbol{T}, \boldsymbol{T}) - a(\boldsymbol{P}, \boldsymbol{P})|. \tag{18}$$

Note that (18) is valid for any kind of approximate solution $\boldsymbol{P}$. It is also convenient to introduce the notion of the *relative energy* (*error*) *norm* or *precision* η_E of the strain energy as

$$\eta_E^2 = \frac{\|e\|_E^2}{a(\boldsymbol{T}, \boldsymbol{T})}. \tag{19}$$

6.1 *Finite element energy norm*

In the special case of the finite element solution when $\boldsymbol{P} = \boldsymbol{T}_h(\boldsymbol{u}_h)$, note (9),

$$\|e\|_{0Eh}^2 = a(\boldsymbol{T},\boldsymbol{T}) - a(\boldsymbol{T}_h,\boldsymbol{T}_h). \tag{20}$$

Moreover, due to (8) and (10)

$$\|e\|_{Eh}^2 = \|e\|_{0Eh}^2 = a(\boldsymbol{T}-\boldsymbol{T}_h,\boldsymbol{T}-\boldsymbol{T}_h). \tag{21}$$

At this point it should be noted that (21) and (20) are equivalent *iff* $\boldsymbol{u}_h$ is the projection of $\boldsymbol{u}$ onto V_h (7) and, at variance with (18), are valid only in this special case. The precision of the finite element strain energy can be determined from expressions

$$\eta_{0Eh}^2 = \frac{\|e\|_{0Eh}^2}{a(\boldsymbol{T},\boldsymbol{T})} \quad ; \quad \eta_{Eh}^2 = \frac{\|e\|_{Eh}^2}{a(\boldsymbol{T},\boldsymbol{T})}. \tag{22}$$

6.2 *Energy norm of the smoothed finite element solution*

The error in energy of the smoothed finite element solution $\boldsymbol{S}_h \in Q_h$ referred to the original (raw) solution $\boldsymbol{T}_h(\boldsymbol{u}_h)$ can be defined as the difference between the energies of raw $\boldsymbol{T}_h$ and smoothed $\boldsymbol{S}_h$ solutions,

$$e_{EhS} = a(\boldsymbol{T}_h,\boldsymbol{T}_h) - a(\boldsymbol{S}_h,\boldsymbol{S}_h). \tag{23}$$

The corresponding relative error will be

$$H_{EhS} = \frac{e_{EhS}}{a(\boldsymbol{T}_h,\boldsymbol{T}_h)}. \tag{24}$$

The appropriate energy norm will be given, in the accordance with (18), by

$$\|e\|_{EhS}^2 = \left|a(\boldsymbol{T}_h,\boldsymbol{T}_h) - a(\boldsymbol{S}_h,\boldsymbol{S}_h)\right|. \tag{25}$$

Consequently, the precision follows from

$$\eta_{EhS}^2 = \left|H_{Ehs}\right|. \tag{26}$$

If we consider averaged solution, $\boldsymbol{S}_h = \boldsymbol{T}_h^A$, the appropriate expressions can be obtained from (23, 25 etc.) by the simple replacement of $\boldsymbol{S}_h$ by $\boldsymbol{T}_h^A$, and/or of indices S by A.

In the special case when $\boldsymbol{S}_h = \boldsymbol{T}_h^L$ is a *projection* of the raw finite element solution $\boldsymbol{T}_h$ (13), one can write that

$$\|e\|_{0EhL}^2 = a(\boldsymbol{T}_h,\boldsymbol{T}_h) - a(\boldsymbol{T}_h^L,\boldsymbol{T}_h^L) \tag{27}$$

and

$$\|e\|_{EhL}^2 = a(\boldsymbol{T}_h-\boldsymbol{T}_h^L,\boldsymbol{T}_h-\boldsymbol{T}_h^L). \tag{28}$$

The relative energy (error) norm or precision η_{EhL}^2 of the strain energy as the energy of error of the projected $\boldsymbol{T}_h^L$ with respect to the raw finite element solution $\boldsymbol{T}_h$ follows from

$$\eta_{EhL}^2 = \frac{\|e\|_{EhL}^2}{a(\boldsymbol{T}_h,\boldsymbol{T}_h)}. \tag{29}$$

7. ZIENKIEWICZ–ZHU ERROR ESTIMATOR

On the basis of the exhaustive computational evidence, Zienkiewicz and Zhu [4] concluded that (28) is numerically close to (20), at least for the low order elements. The motivation and plausible explanation of this fact has been numerical closeness of the exact $\boldsymbol{T}$ and projected finite element solution $\boldsymbol{T}_h^L$. Moreover, it has been soon recognized that *for the error estimates*, global projection procedure is unnecessary, if not contraproductive, and simple averaging or more or less equivalent local procedures are sufficient [5,6]. Analytical validation of Z–Z (error estimators based on recovery techniques), based mainly on their equivalence with residual estimators, is also available [7,8]. Anyhow, Z–Z type error estimators can be defined by the expression

$$\|e\|_{Z-Z}^2 = a(\boldsymbol{T}_h - \boldsymbol{S}_h, \boldsymbol{T}_h - \boldsymbol{S}_h), \tag{30}$$

and the hypothesis of Zienkiewicz and Zhu is that

$$\|e\|_{Eh}^2 \approx \|e\|_{Z-Z}^2 . \tag{31}$$

Analogously to (29) one can also introduce the notion of the relative value of error estimator

$$\eta_{Z-Z}^2 = \frac{\|e\|_{Z-Z}^2}{a(\boldsymbol{T}_h, \boldsymbol{T}_h)}, \tag{32}$$

and the relationship similar to (31):

$$\eta_{Eh}^2 \approx \eta_{Z-Z}^2 \tag{33}$$

Note that, due to (9), (33) is not only more meaningful, but also more reliable estimate of error, because ' . . . , *it would be desirable to overestimate rather than underestimate the exact error* . . .' [6].

8. NUMERICAL EXAMPLE

The problem of the rectangular in–plane loaded plate with the prescribed displacements is borrowed from [9]. Rectangular domain determined by the points (0,0), (2,0), (2,1) and (0,1) is considered. Modulus of elasticity is E = 1 and Poisson's coefficient $\nu = 0$. Analytical displacements are given by the relationship:

$$u = v = x\left(x - \frac{2}{3}\right)\left(x - \frac{3}{2}\right)(x-2)(1+y) + 10y\left(y - \frac{1}{3}\right)\left(y - \frac{3}{4}\right)(y-1)(1+x). \tag{34}$$

Exact strains and stresses are calculated from the expressions of elasticity. For the plate thickness 1, the total strain energy is 5.29563. From the Fig. 1. it is evident that raw finite element solution $\boldsymbol{T}_h$ is superior over the projected solution $\boldsymbol{T}_h^L$, as long as strain energies are considered. This behavior already should be expected due to the inequality (15). Smoothed solution obtained by simple averaging of nodal stress values, $\boldsymbol{S}_h = \boldsymbol{T}_h^A$, is even worse, as it should be expected on the basis of (16). Of course, one can ask the question why to make smoothing at all – if the result (in the strain energy) is always worse than raw finite element solution. However, the smoothed solutions, although accompanied with a larger overall (energy) error, have this error more evenly distributed and hence, as a rule, nodal errors of smoothed derivatives are smaller.

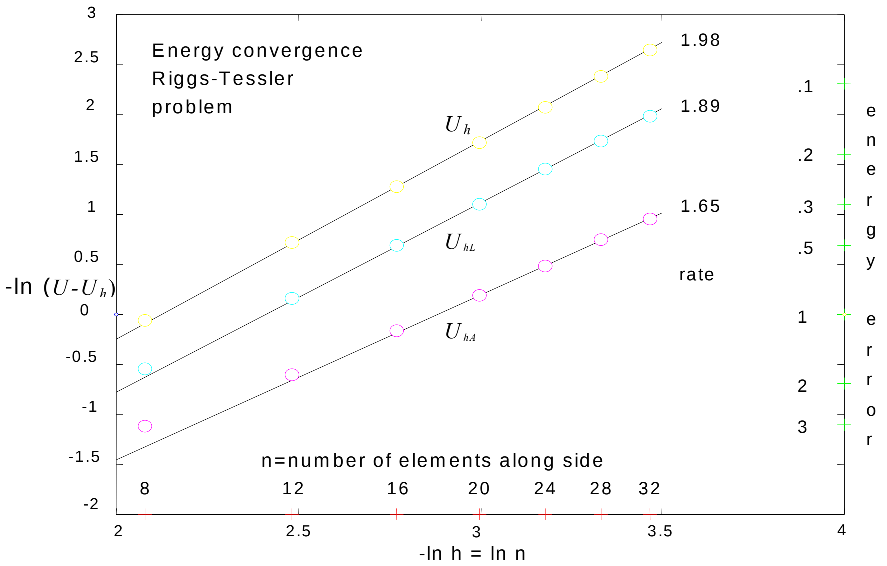

Fig. 1. Convergence of energy

Afterwards, the smoothed solutions are more ‘user friendly’, because allow a construction of a better, more realistic, smooth visual model of the stress state for the problem under consideration.

On the next graph, Fig. 2., the convergence of the error norms for the aforementioned models is shown. It is evident that, the energy of error of the finite element solution (20) is equal to the error in energy (21). In this and in subsequent figures the relative *percentage* values of energy error norms are shown, (expressed per cent of the theoretical value of strain energy), i.e. $\eta \times 100$ per cent. However, such equivalence does not appear when smoothed stress fields are considered, because these fields are *not* orthogonal projections of the exact solution onto the finite element subspace. As it should be expected on the basis of the Fig. 1., the *errors of energy* η_{EL} and η_{EA} of both the projected $\boldsymbol{T}_h^L$ and averaged $\boldsymbol{T}_h^A$ respectively replacing $\boldsymbol{P}$ in (18), are evidently larger than the error in energy of the raw finite element solution (21). In contrary, the *energies in error* for these solutions, η_{REA} and η_{REL}, both based on

$$\eta_{RES}^2 = a(\boldsymbol{T}-\boldsymbol{S}_h, \boldsymbol{T}-\boldsymbol{S}_h)/a(\boldsymbol{T},\boldsymbol{T}) \qquad (35)$$

with $\boldsymbol{S}_h$ replaced by $\boldsymbol{T}_h^A$ and $\boldsymbol{T}_h^L$ respectively, are unexpectedly smaller than $\eta_{Eh} = \eta_{OEH}$ (22) for the raw finite element solution.

Of course, the question remains why the solution numerically closer to the exact one, is worse in strain energy? A physical explanation can be that a smoothed solution has larger error in satisfying the essential equations of a problem (1) and (2) than the finite element solution. Finally, it is interesting to note that graphs of the type η_{REA} and η_{REL} are often used as proofs of the superiority of various recovery procedures [5,10,11].

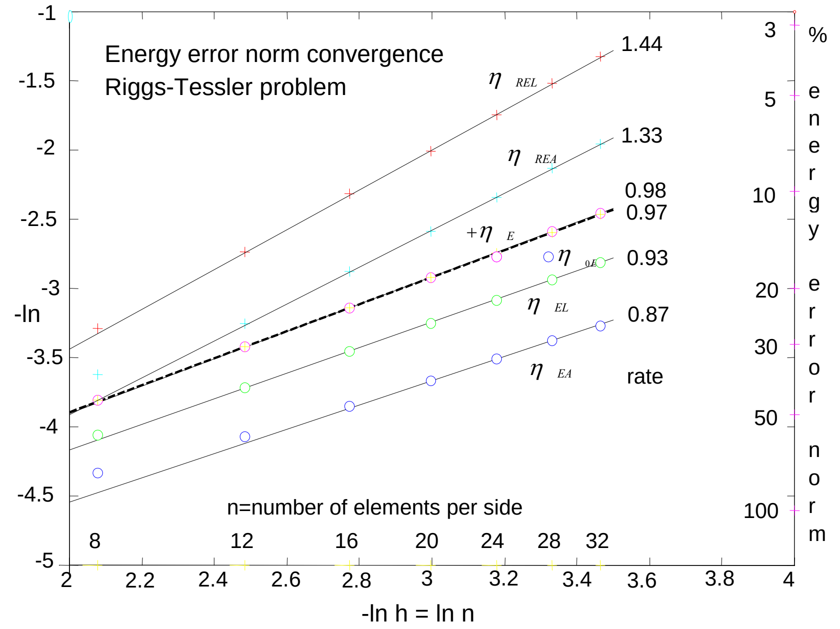

Fig. 2. Convergence of energy error norms

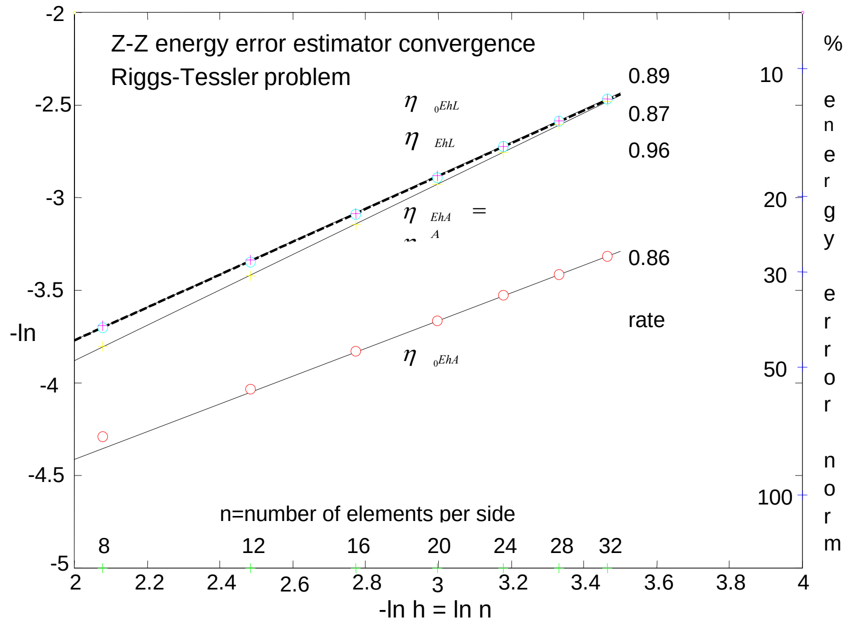

Fig. 3. Convergence of Z–Z energy error estimators

On the Fig. 3., the Z–Z energy error estimators (32) based on averaged and projected stress fields (in fact, the energies of error of these solutions with respect to the finite element solution) i.e. η^{A}_{Z-Z} and η^{L}_{Z-Z} respectively, are compared with the corresponding errors in energies η_{OEhL} (29) and η_{0EhA} based on (26) with $\boldsymbol{S}_h$ replaced by $\boldsymbol{T}_h^{A}$. First of all, it is interesting to note that η^{2}_{EhL} and η^{2}_{0EhL} practically overlap – because $\boldsymbol{T}_h^{L}$ is a projection of the finite element solution $\boldsymbol{T}_h$ onto the continuous finite element space (see (14)). In contrary, for the averaged solution $\boldsymbol{T}_h^{A}$ the energy of error with respect to the raw finite element solution η^{A}_{Z-Z} and the corresponding error in energy η_{0EhA} are entirely different, either in the accuracy and in the convergence rate.

Nevertheless, the values of Z–Z error estimators η^{A}_{Z-Z} and η^{L}_{Z-Z} for the averaged and projected solutions respectively are very close. Hence, one can conclude that the calculation of Z–Z estimators is a robust procedure, not only with respect to the choice of sampling points [8], but also with respect to the choice of the smoothing procedure.

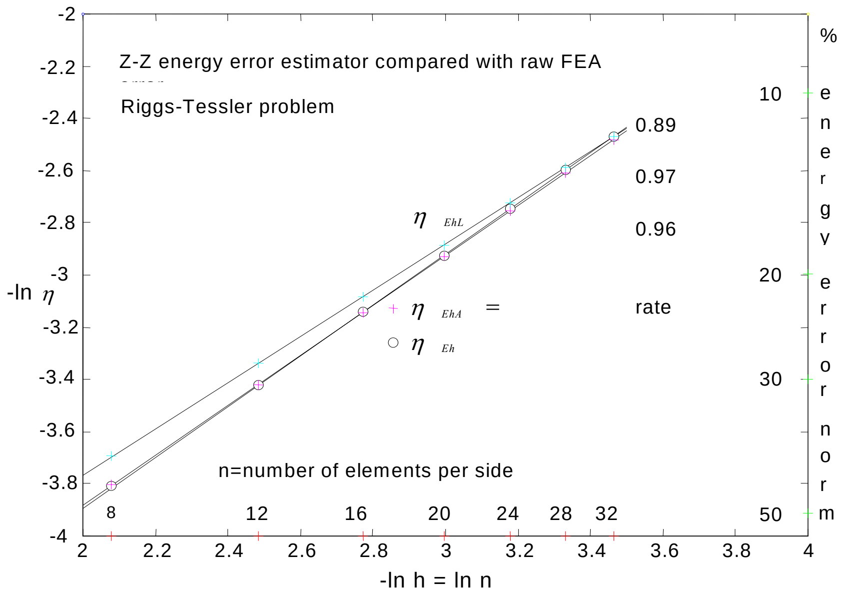

Fig. 4. Comparison of Z–Z energy error estimators and raw finite element error

On the Fig. 4. we can see that the relative error of the finite element solution, $\eta_{OE\eta}$ or η_{Eh} (22), is extremely well represented by η^{A}_{Z-Z}, i.e. by *Z–Z* error estimator based on the averaged solution, while *Z–Z* error estimator based on the on the projected values η^{L}_{Z-Z} also converges asymptotically to the relative error of the finite element solution. Hence, one can conclude that, as long as error estimates are our concern, it is sufficient, and can be recommended, to use averaged solutions, i.e. η^{A}_{Z-Z} estimator.

9. CONCLUSIONS

It is presently a common opinion that projected, or even arbitrarily smoothed stress fields are 'more accurate' than raw finite element solution. However, in the present paper it has been shown, either analytically and numerically, that the strain energy of the projected (i.e. the best approximate for the given finite element space) solution is necessarily more in error than that of finite element solution. In this paper also a minor but useful improvement of Z–Z error estimator is proposed, i.e. the use of its relative η_{Z-Z} instead of absolute $\|e\|_{Z\text{-}Z}$ value.

A practical recommendation based upon the present paper is that for the error estimates, and their eventual use for the adaptive remeshing, simple averaging, or eventually similar local procedures, are sufficient. This conclusion is also in the accordance with the results of some other researchers [6]. However, global projection is preferable as a smoothing (postprocessing) procedure, at least in the comparison with simple averaging [12].

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